

CONFIGURATION SPACES OF TORI

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ABSTRACT. The configuration spaces $\mathcal{C}^n(\mathbb{T}^2) = \{Q \subset \mathbb{T}^2 \mid \#Q = n\}$ and $\mathcal{E}^n(\mathbb{T}^2) = \{(q_1, \dots, q_n) \in (\mathbb{T}^2)^n \mid q_i \neq q_j \ \forall i \neq j\}$ of a torus $\mathbb{T}^2 = \mathbb{C}/\{\text{a lattice}\}$ are complex manifolds. We prove that for $n > 4$ any holomorphic self-map F of $\mathcal{C}^n(\mathbb{T}^2)$ either carries the whole of $\mathcal{C}^n(\mathbb{T}^2)$ into an orbit of the diagonal $\text{Aut}(\mathbb{T}^2)$ action in $\mathcal{C}^n(\mathbb{T}^2)$ or is of the form $F(Q) = T(Q)Q$, where $T: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \text{Aut}(\mathbb{T}^2)$ is a holomorphic map. We also prove that for $n > 4$ any endomorphism of the torus braid group $B_n(\mathbb{T}^2) = \pi_1(\mathcal{C}^n(\mathbb{T}^2))$ with a non-abelian image preserves the pure torus braid group $PB_n(\mathbb{T}^2) = \pi_1(\mathcal{E}^n(\mathbb{T}^2))$.

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1. INTRODUCTION

In this paper we study certain properties of the torus configuration spaces and torus braid groups.

1.1. Configuration spaces. The n^{th} configuration space $\mathcal{C}^n = \mathcal{C}^n(X)$ of a complex space X consists of all n point subsets (“configurations”) $Q = \{q_1, \dots, q_n\} \subset X$. An automorphism $T \in \text{Aut } X$ gives rise to the holomorphic endomorphism F_T of \mathcal{C}^n defined by $F_T(Q) = TQ = \{Tq_1, \dots, Tq_n\}$. If $\text{Aut } X$ is a complex Lie group, one may take $T = T(Q)$ depending analytically on a configuration $Q \subset X$ and define the corresponding holomorphic endomorphism F_T by $F_T(Q) = T(Q)Q$. Such a map F_T is called *tame*. On the other hand, choosing a base configuration $Q^0 = \{q_1^0, \dots, q_n^0\} \subset X$, one may define an endomorphism F_{T, Q^0} , $F_{T, Q^0}(Q) = T(Q)Q^0 = \{T(Q)q_1^0, \dots, T(Q)q_n^0\}$, which certainly maps the whole configuration space into one orbit $(\text{Aut } X)Q^0$ of the diagonal $\text{Aut } X$ action in \mathcal{C}^n ; endomorphisms that have the latter property are said to be *orbit-like*.

Of course, configuration spaces of a certain specific space X may admit “sporadic” holomorphic endomorphisms, which are neither tame nor orbit-like; but it is not to be expected that there is a general construction of such endomorphisms.

Let us restrict ourselves to the simplest interesting case when X is a non-hyperbolic Riemann surface¹, i.e. one of the following curves: complex projective line \mathbb{CP}^1 ; complex affine line \mathbb{C} ; complex affine line with one puncture $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$; a torus $\mathbb{T}^2 = \mathbb{C}/\{\text{a lattice of rank } 2\}$.

V. Lin (see [1, 2, 3], etc.) proved that for $n > 4$ and $X = \mathbb{C}$ or $X = \mathbb{CP}^1$ every holomorphic endomorphism F of $\mathcal{C}^n(X)$ is either tame or orbit-like. Moreover, F is tame if and only if it is non-abelian, meaning that the image $F_*(\pi_1(\mathcal{C}^n(X)))$ of the induced endomorphism F_* of the fundamental group $\pi_1(\mathcal{C}^n(X))$ (which is the braid group $B_n(X)$ of X) is a non-abelian group; otherwise F is orbit-like. Similar results were obtained by V. Zinde (see [4, 5, 6, 7, 8]) for $X = \mathbb{C}^*$.

In this paper, we complete the story for all non-hyperbolic Riemann surfaces; one of our main results is as follows.

THEOREM 1.1. *For $n > 4$, each holomorphic map $F: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ is either tame or orbit-like. In particular, any automorphism of $\mathcal{C}^n(\mathbb{T}^2)$ is tame.*

Moreover, F is tame if and only if the induced endomorphism F_ of the fundamental group $\pi_1(\mathcal{C}^n(\mathbb{T}^2))$ is non-abelian, i.e. its image $F_*(\pi_1(\mathcal{C}^n(\mathbb{T}^2)))$ is a non-abelian group.*

This theorem has two immediate corollaries. The first shows that non-abelian holomorphic endomorphisms of $\mathcal{C}^n(\mathbb{T}^2)$ admit a simple classification up to a holomorphic homotopy.²

COROLLARY 1.2. *For $n > 4$, the set $\mathcal{H}(\mathcal{C}^n(\mathbb{T}^2), \mathcal{C}^n(\mathbb{T}^2))$ of all holomorphic homotopy classes of non-abelian holomorphic endomorphisms of $\mathcal{C}^n(\mathbb{T}^2)$ is in a natural one-to-one correspondence with the set $\mathcal{H}(\mathcal{C}^n(\mathbb{T}^2), \text{Aut } \mathbb{T}^2)$ of all holomorphic homotopy classes of holomorphic maps $\mathcal{C}^n(\mathbb{T}^2) \rightarrow \text{Aut } \mathbb{T}^2$.*

¹The automorphism group of a hyperbolic Riemann surface X is discrete; in fact, a “generic” hyperbolic Riemann surface does not admit a non-identical automorphism.

²That is, a homotopy within the space of holomorphic mappings.

COROLLARY 1.3. *Let $n > 4$ and $G = \text{Aut } \mathcal{C}^n(\mathbb{T}^2)$. Then the orbits of the natural G -action in $\mathcal{C}^n(\mathbb{T}^2)$ coincide with the orbits of the diagonal $(\text{Aut } \mathbb{T}^2)$ -action in $\mathcal{C}^n(\mathbb{T}^2)$.*

Our second main theorem deals with the torus braid group $B_n(\mathbb{T}^2) = \pi_1(\mathcal{C}^n(\mathbb{T}^2))$ and its pure braid subgroup $P_n(\mathbb{T}^2)$, which is the fundamental group of the *ordered* configuration space

$$\mathcal{E}^n(\mathbb{T}^2) = \{q = (q_1, \dots, q_n) \in (\mathbb{T}^2)^n \mid q_i \neq q_j \ \forall i \neq j\}.$$

THEOREM 1.4. *a) Let $n > 4$. Then the pure braid group $P_n(\mathbb{T}^2) \subset B_n(\mathbb{T}^2)$ is invariant under any non-abelian endomorphism φ of the whole braid group $B_n(\mathbb{T}^2)$, that is, $\varphi(P_n(\mathbb{T}^2)) \subseteq P_n(\mathbb{T}^2)$.*

b) Let $n > 4$ and $n > m$. Then any homomorphism $\varphi: B_n(\mathbb{T}^2) \rightarrow B_m(\mathbb{T}^2)$ is abelian, i.e. $\varphi(B_n(\mathbb{T}^2))$ is an abelian group.

For automorphisms of the classical Artin braid group $B_n(\mathbb{C})$, an analogue of part (a) was proved by Artin himself in 1947 (see [9]). V. Lin [10, 2, 11, 12, 3] generalized this result of Artin to non-abelian endomorphisms of $B_n(\mathbb{C})$ and $B_n(\mathbb{CP}^1)$, respectively; the case of $B_n(\mathbb{C}^*)$ was handled by V. Zinde [7, 8]. In 1992, N. Ivanov [13] proved a similar result for *automorphisms* of the braid group $B_n(X)$ of any finite type Riemann surface but for $X = \mathbb{CP}^1$. Theorem 1.4(a) completes the story for non-hyperbolic curves. Analogues of statement (b) were known for the braid groups $B_n(\mathbb{C})$, $B_n(\mathbb{CP}^1)$ and $B_n(\mathbb{C}^*)$ (see papers by V. Lin and V. Zinde quoted above)

1.2. Plan of the proof of Theorem 1.1. First, due to Theorem 1.4(a), every continuous non-abelian self-map F fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{E}^n(\mathbb{T}^2) & \xrightarrow{f} & \mathcal{E}^n(\mathbb{T}^2) \\ p \downarrow & & \downarrow p \\ \mathcal{C}^n(\mathbb{T}^2) & \xrightarrow{F} & \mathcal{C}^n(\mathbb{T}^2), \end{array} \quad (1.1)$$

where p is the natural projection

$$\mathcal{E}^n(\mathbb{T}^2) \ni q = (q_1, \dots, q_n) \mapsto \{q_1, \dots, q_n\} = Q \in \mathcal{C}^n(\mathbb{T}^2). \quad (1.2)$$

The map f is *strictly equivariant* with respect to the standard action of the symmetric group $\mathbf{S}(n)$ in $\mathcal{E}^n(\mathbb{T}^2)$, meaning that there is an automorphism α of $\mathbf{S}(n)$ such that

$$F(\sigma q) = \alpha(\sigma)F(q) \quad \text{for all } q \in \mathcal{E}^n(\mathbb{T}^2) \text{ and } \sigma \in \mathbf{S}(n).$$

Moreover, f is non-constant and holomorphic whenever F is so.

A torus $\mathbb{T}^2 = \mathbb{C}/\{\text{a lattice of rank 2}\}$ which we deal with is an additive complex Lie group. To study non-constant strictly equivariant holomorphic endomorphisms f of the space $\mathcal{E}^n = \mathcal{E}^n(\mathbb{T}^2)$, we start with an explicit description of all non-constant holomorphic maps $\lambda: \mathcal{E}^n \rightarrow \mathbb{T}^2 \setminus \{0\}$. The set L of all such maps is finite and separates points of a certain submanifold $M \subset \mathcal{E}^n$ of complex codimension 1. An endomorphism f induces a self-map f^* of L via $f^*: L \ni \lambda \mapsto f^*\lambda = \lambda \circ f \in L$.

The map f^* carries important information about f . In order to investigate behaviour of f^* and then recover this information, we endow L with the following simplicial structure: a subset $\Delta^s = \{\lambda_0, \dots, \lambda_s\} \subseteq L$ is said to be an *s-simplex* whenever $\lambda_i - \lambda_j \in L$ for all distinct i, j . The action of $\mathbf{S}(n)$ in \mathcal{E}^n induces a simplicial $\mathbf{S}(n)$ -action in the complex L ; the orbits of this action may be exhibited

explicitly. On the other hand, the map $f^*: L \rightarrow L$ defined above is simplicial and preserves dimension of simplices; since f is equivariant, f^* is nicely related to the $\mathbf{S}(n)$ action on L .

Studying all these things together, we eventually come to the conclusion that f is *tame*, meaning that there exists an $\mathbf{S}(n)$ invariant holomorphic map

$$t: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \text{Aut } \mathbb{T}^2 \quad (1.3)$$

and a permutation $\sigma \in \mathbf{S}(n)$ such that

$$f(q) = \sigma t(q)q = (t(q)q_{\sigma^{-1}(1)}, \dots, t(q)q_{\sigma^{-1}(n)}) \quad \forall q = (q_1, \dots, q_n) \in \mathcal{E}^n(\mathbb{T}^2). \quad (1.4)$$

The latter formula implies that the original endomorphism F is tame.

For the complete proofs, see Section 2 (the proof of Theorem 1.4 and some related algebraic results), Section 3 (a simplicial complex of holomorphic maps $\mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathbb{T}^2 \setminus \{0\}$) and Section 4 (the proof of Theorem 1.1).

1.3. Notation and definitions. For the reader's convenience, we collected here the main notation and definitions used throughout the paper.

DEFINITION 1.5. The *ordered* and *non-ordered configuration spaces* of a torus \mathbb{T}^2 are defined as follows:

$$\begin{aligned} \mathcal{E}^n(\mathbb{T}^2) &= \{(q_1, \dots, q_n) \in (\mathbb{T}^2)^n \mid q_i \neq q_j \ \forall i \neq j\}, \\ \mathcal{C}^n(\mathbb{T}^2) &= \{Q \subset \mathbb{T}^2 \mid \#Q = n\}. \end{aligned}$$

The projection

$$p: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2), \quad p(q) = p(q_1, \dots, q_n) = \{q_1, \dots, q_n\} = Q,$$

is an $\mathbf{S}(n)$ Galois covering and we have the exact sequence of the corresponding fundamental groups

$$1 \rightarrow \pi_1(\mathcal{E}^n(\mathbb{T}^2)) \xrightarrow{p_*} \pi_1(\mathcal{C}^n(\mathbb{T}^2)) \rightarrow \mathbf{S}(n) \rightarrow 1. \quad (1.5)$$

The fundamental group $\pi_1(\mathcal{C}^n(\mathbb{T}^2))$ is called the *torus braid group* and is denoted by $B_n(\mathbb{T}^2)$. Its normal subgroup $P_n(\mathbb{T}^2) \stackrel{\text{def}}{=} \pi_1(\mathcal{E}^n(\mathbb{T}^2))$ is called the *pure torus braid group*.

DEFINITION 1.6. A group homomorphism $\varphi: G \rightarrow H$ is said to be *abelian* if its image $\varphi(G)$ is an abelian subgroup of H . A continuous map $F: X \rightarrow Y$ of arcwise-connected spaces is called *abelian* if the induced homomorphism $F_*: \pi_1(X) \rightarrow \pi_1(Y)$ of the fundamental groups is abelian.

For a complex space X , we denote by $\text{Aut } X$ the group of all biholomorphic automorphisms of X . For algebraic X , $\text{Aut}_{\text{reg}} X$ stands for the group of all biregular automorphisms.

The group $\text{Aut } \mathbb{T}^2 = \text{Aut}_{\text{reg}} \mathbb{T}^2$ is a compact complex Lie group isomorphic to a semidirect product $\mathbb{T}^2 \rtimes \mathbb{Z}_k$; here $k = 2$ if the torus \mathbb{T}^2 has no complex multiplications; otherwise, either $k = 4$ or $k = 6$.

DEFINITION 1.7. A holomorphic endomorphism F of $\mathcal{C}^n(\mathbb{T}^2)$ is said to be *tame* if there is a holomorphic map $T: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \text{Aut } \mathbb{T}^2$ such that $F(Q) = T(Q)Q$ for all $Q \in \mathcal{C}^n(\mathbb{T}^2)$.

DEFINITION 1.8. A holomorphic map $F: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ is said to be *orbit-like* if its image $F(\mathcal{C}^n(\mathbb{T}^2))$ is contained in one orbit of the diagonal $\text{Aut } \mathbb{T}^2$ action in $\mathcal{C}^n(\mathbb{T}^2)$.

2. SOME ALGEBRAIC PROPERTIES OF TORUS BRAID GROUPS

The main aim of this section is to prove Theorem 1.4. To this end, we first need to prove some auxiliary results, which also seem to be of independent interest.

2.1. Zariski presentation and homomorphism $B_n \rightarrow B_n(\mathbb{T}^2)$. We will use the following presentation of the torus braid group $B_n(\mathbb{T}^2)$ found by O. Zariski [14]³.

Generators:

$$\sigma_1, \dots, \sigma_{n-1}; a_1, a_2; \quad (2.1)$$

relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2, \ i, j = 1, \dots, n - 3; \quad (2.2)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \dots, n - 2; \quad (2.3)$$

$$\sigma_i a_k = a_k \sigma_i \quad \text{for } k = 1, 2 \text{ and } i = 2, \dots, n - 1; \quad (2.4)$$

$$(\sigma_1^{-1} a_k)^2 = (a_k \sigma_1^{-1})^2 \quad \text{for } k = 1, 2; \quad (2.5)$$

$$\sigma_1 \dots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \dots \sigma_1 = a_1 a_2^{-1} a_1^{-1} a_2; \quad (2.6)$$

$$a_2 \sigma_1^{-1} a_1^{-1} \sigma_1 a_2^{-1} \sigma_1^{-1} a_1 \sigma_1 = \sigma_1^2. \quad (2.7)$$

2.2. Normal series in the torus pure braid groups $P_n(\mathbb{T}^2)$. The exact homotopy sequence (1.5) of the covering $p: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ may be written as

$$1 \rightarrow P_n(\mathbb{T}^2) \xrightarrow{p_*} B_n(\mathbb{T}^2) \xrightarrow{\mu} \mathbf{S}(n) \rightarrow 1, \quad (2.8)$$

where the epimorphism μ is defined by

$$\mu(\sigma_i) = (i, i + 1) \text{ for } i = 1, \dots, n - 1 \text{ and } \mu(a_1) = \mu(a_2) = 1. \quad (2.9)$$

Let us take a point $(c_1, \dots, c_n) \in \mathcal{E}^n(\mathbb{T}^2)$ and for each $m = 1, \dots, n - 1$ consider the ordered configuration space $\mathcal{E}^{n-m}(\mathbb{T}^2 \setminus \{c_1, \dots, c_m\})$ of the torus \mathbb{T}^2 punctured at m points c_1, \dots, c_m ; a point $q \in \mathcal{E}^{n-m}(\mathbb{T}^2 \setminus \{c_1, \dots, c_m\})$ has $n - m$ pairwise distinct components $q_{m+1}, \dots, q_n \in \mathbb{T}^2 \setminus \{c_1, \dots, c_m\}$. Each of these configuration spaces is an Eilenberg–MacLane $K(\pi, 1)$ space for its fundamental group π (see E. Fadell and L. Neuwirth [16], Corollary 2.2), which is the pure braid group (on $n - m$ strands) $P_{n-m;m}$ of the punctured torus $\mathbb{T}^2 \setminus \{c_1, \dots, c_m\}$. The projections

$$t_1: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathbb{T}^2, \ (q_1, \dots, q_n) \mapsto q_1, \quad (2.10)$$

$$t_{m+1}: \mathcal{E}^{n-m}(\mathbb{T}^2 \setminus \{c_1, \dots, c_m\}) \rightarrow \mathbb{T}^2 \setminus \{c_1, \dots, c_m\}, \quad (2.11)$$

$$(q_{m+1}, \dots, q_n) \mapsto q_{m+1}, \ 1 \leq m \leq n - 2,$$

define locally trivial, smooth fibrings⁴ with fibres isomorphic to $\mathcal{E}^{n-1}(\mathbb{T}^2 \setminus \{c_1\})$ and $\mathcal{E}^{n-m-1}(\mathbb{T}^2 \setminus \{c_1, \dots, c_{m+1}\})$, respectively. The final segments of the exact homotopy sequences of these fibrings look, respectively, as

$$1 \rightarrow P_{n-1;1} \rightarrow P_n(\mathbb{T}^2) \rightarrow \mathbb{Z}^2 \rightarrow 1$$

³Another presentation was found by J. Birman [15].

⁴See [16] for the case of arbitrary Riemann surfaces.

and

$$1 \rightarrow P_{n-m-1;m+1} \rightarrow P_{n-m;m} \rightarrow \mathbb{F}_m \rightarrow 1,$$

where \mathbb{F}_m stands for a free group of rank m . This leads immediately to the following statement, which is an analogue of a well-known Markov theorem for Artin pure braid groups, proved in [17].

PROPOSITION 2.1. *The subgroups $P_{n-s;s}$ fit into the normal series:*

$$\{1\} \subset P_{1;n-1} \subset \cdots \subset P_{n-m-1;m+1} \subset P_{n-m;m} \subset \cdots \subset P_{n-1;1} \subset P_{n;0} = P_n(\mathbb{T}^2)$$

with the factors

$$\begin{aligned} P_{1;n-1}/\{1\} &= P_{1;n-1} = \pi_1(\mathbb{T}^2 \setminus \{a_1, \dots, a_{n-1}\}) \cong \mathbb{F}_{n-1}, \dots, \\ P_{n-m;m}/P_{n-m-1;m+1} &\cong \mathbb{F}_m, \dots, \\ P_{n-1;1}/P_{n-2;2} &\cong \mathbb{F}_2, \quad P_n(\mathbb{T}^2)/P_{n-1;1} \cong \mathbb{Z}^2. \end{aligned} \quad (2.12)$$

In particular, any non-trivial subgroup $H \subseteq P_n(\mathbb{T}^2)$ admits a non-trivial homomorphism to \mathbb{Z} .

The last sentence of the proposition implies:

PROPOSITION 2.2. *A group G with finite abelianization $G/G' = G/[G, G]$ has no non-trivial homomorphisms to the pure torus braid group $P_n(\mathbb{T}^2)$.*

Recall that the Artin braid group $B_n = \pi_1(\mathcal{C}^n(\mathbb{C}))$ has generators $\sigma_1, \dots, \sigma_{n-1}$ and the system of defining relations (2.2), (2.3). Thus, we have a natural homomorphism $i: B_n \rightarrow B_n(\mathbb{T}^2)$ sending all generators σ 's of B_n to the elements of the same name in $B_n(\mathbb{T}^2)$ (according to N. Ivanov [13], i is injective; however, we will not use this fact).

LEMMA 2.3. *Let $n > 4$ and let $\varphi: B_n(\mathbb{T}^2) \rightarrow B_m(\mathbb{T}^2)$ be a homomorphism such that the composition*

$$\Phi = \mu \circ \varphi \circ i: B_n \xrightarrow{i} B_n(\mathbb{T}^2) \xrightarrow{\varphi} B_m(\mathbb{T}^2) \xrightarrow{\mu} \mathbf{S}(m) \quad (2.13)$$

is an abelian homomorphism. Then φ is abelian. In particular, φ is abelian whenever $\mu \circ \varphi$ is so.

Proof. Let $\Phi': B'_n \rightarrow \mathbf{S}(m)$ denote the restriction of Φ to the commutator subgroup B'_n of the group B_n . Since Φ is abelian, Φ' is trivial and hence $\varphi(i(B'_n)) \subseteq \text{Ker } \mu = P_m(\mathbb{T}^2)$. By Gorin–Lin Theorem [18], for $n > 4$ the group B'_n is perfect⁵, and Proposition 2.2 shows that $\varphi(i(B'_n)) = 1$. Hence $\varphi \circ i$ is abelian, and relations $\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}$ imply $\varphi(i(\sigma_1)) = \dots = \varphi(i(\sigma_{n-1}))$. By definition, i sends the generators $\sigma_1, \dots, \sigma_{n-1}$ of B_n into the corresponding generators $\sigma_1, \dots, \sigma_{n-1}$ of $B_n(\mathbb{T}^2)$; thus, the latter ones satisfy

$$\varphi(\sigma_1) = \dots = \varphi(\sigma_{n-1}). \quad (2.14)$$

Together with (2.4) this shows that

$$\varphi(a_k) \varphi(\sigma_j) = \varphi(\sigma_j) \varphi(a_k) \quad \text{for all } j, k. \quad (2.15)$$

By (2.14) and (2.6),

$$(\varphi(\sigma_1))^{2(n-1)} = \varphi(a_1) \varphi(a_2)^{-1} \varphi(a_1)^{-1} \varphi(a_2), \quad (2.16)$$

⁵That is, B'_n coincides with its commutator subgroup $B''_n = [B'_n, B'_n]$.

and by (2.15) and (2.7),

$$\varphi(a_2)\varphi(a_1)^{-1}\varphi(a_2)^{-1}\varphi(a_1) = \varphi(\sigma_1)^2. \quad (2.17)$$

Relations (2.15) and (2.17) imply

$$\varphi(\sigma_1)^2 = \varphi(a_2)^{-1}\varphi(a_1)\varphi(a_2)\varphi(a_1)^{-1}. \quad (2.18)$$

Multiplying (2.16) and (2.18) we obtain

$$1 = (\varphi(\sigma_1))^{2(n-1)}(\varphi(\sigma_1))^2 = (\varphi(\sigma_1))^{2n}.$$

However, by E. Fadell and L. Neuwirth [16], Theorem 8, the torus braid group $B_m(\mathbb{T}^2)$ has no elements of a finite order, this implies $\varphi(\sigma_1) = 1$. Thus

$$\varphi(a_2)^{-1}\varphi(a_1)\varphi(a_2)\varphi(a_1)^{-1} = 1,$$

and φ is an abelian homomorphism. \square

2.3. Proof of Theorem 1.4. Let $n > 4$ and let φ be a non-abelian endomorphism of $B_n(\mathbb{T}^2)$. By Lemma 2.3, the homomorphism $\Phi = \mu \circ \varphi \circ i: B_n \rightarrow \mathbf{S}(n)$ in (2.13) is non-abelian. Thus, by V. Lin's theorem (see [2], Section 4; a complete proof may be found in [13, 11, 12]), Φ coincides with the standard epimorphism $B_n \rightarrow \mathbf{S}(n)$ up to an automorphism of $\mathbf{S}(n)$. It follows that the homomorphism $\mu \circ \varphi: B_n(\mathbb{T}^2) \rightarrow \mathbf{S}(n)$ is surjective. N. Ivanov (see [13], Theorem 1) proved that for $n > 4$, any non-abelian homomorphism $B_n(\mathbb{T}^2) \rightarrow \mathbf{S}(n)$ whose image is a primitive permutation group on n letters coincides with the standard epimorphism $\mu: B_n(\mathbb{T}^2) \rightarrow \mathbf{S}(n)$ up to an automorphism of $\mathbf{S}(n)$. Thus, $\text{Ker}(\mu \circ \varphi) = P_n(\mathbb{T}^2) = \text{Ker} \mu$, which implies $\varphi(P_n(\mathbb{T}^2)) \subseteq \text{Ker} \mu = P_n(\mathbb{T}^2)$ thus proving part (a) of the theorem.⁶

To prove part (b), we use another Lin's theorem (see quotations above), which says that for $n > \max(m, 4)$ any homomorphism $B_n \rightarrow \mathbf{S}(m)$ is abelian; it follows that the homomorphism $\Phi = \mu \circ \varphi \circ i: B_n \rightarrow \mathbf{S}(m)$ in (2.13) is abelian. By Lemma 2.3, φ is abelian. \square

We skip the proofs of the next two results, which will not be used in what follows.

THEOREM 2.4. *For $n \geq 5$, the abelianization $B'_n(\mathbb{T}^2)/[B'_n(\mathbb{T}^2), B'_n(\mathbb{T}^2)]$ of the commutator subgroup $B'_n(\mathbb{T}^2)$ of the torus braid group $B_n(\mathbb{T}^2)$ is isomorphic to $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.*

THEOREM 2.5. *Let $3 \leq m \leq n \leq 4$ and let $\varphi: B_n(\mathbb{T}^2) \rightarrow B_m(\mathbb{T}^2)$ be a non-abelian homomorphism such that $\mu \circ \varphi$ is non-abelian. Then $\varphi(P_n(\mathbb{T}^2)) \subseteq P_m(\mathbb{T}^2)$.*

3. ORDERED CONFIGURATION SPACES

Here we establish some analytic properties of ordered configuration spaces.

3.1. A Cartesian product structure in $\mathcal{E}^n(\mathbb{T}^2)$. The torus \mathbb{T}^2 acts in $\mathcal{E}^n(\mathbb{T}^2)$ by translations: $q = (q_1, \dots, q_n) \mapsto (q_1 + t, \dots, q_n + t)$, $t \in \mathbb{T}^2$. Any orbit of this action is isomorphic to \mathbb{T}^2 and intersects the submanifold

$$\tilde{M}_0 = \{q = (q_1, \dots, q_{n-1}, 0) \in \mathcal{E}^n(\mathbb{T}^2)\} \cong \mathcal{E}^{n-1}(\mathbb{T}^2 \setminus \{0\})$$

⁶In fact, we proved a slightly stronger property $\varphi^{-1}(P_n(\mathbb{T}^2)) = P_n(\mathbb{T}^2)$.

at a single point. This gives the Cartesian decomposition $\mathcal{E}^n(\mathbb{T}^2) = \mathbb{T}^2 \times \tilde{M}_0$, defined by the following maps

$$\begin{aligned} \mathcal{E}^n(\mathbb{T}^2) \ni (q_1, \dots, q_n) &\mapsto (q_n, (q_1 - q_n, \dots, q_{n-1} - q_n, 0)) \in \mathbb{T}^2 \times \tilde{M}_0, \\ \mathbb{T}^2 \times \tilde{M}_0 \ni (t, (q_1, \dots, q_{n-1}, 0)) &\mapsto (q_1 + t, \dots, q_{n-1} + t, t) \in \mathcal{E}^n(\mathbb{T}^2). \end{aligned}$$

It is easily seen that \tilde{M}_0 is a non-singular irreducible affine algebraic variety; in particular, it is a Stein manifold, whereas $\mathcal{E}^n(\mathbb{T}^2)$ is not so.

3.2. Holomorphic mappings $\mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathbb{T}^2 \setminus \{0\}$. We continue to execute our plan sketched in Section 1.2 with the following

THEOREM 3.1. *Any non-constant holomorphic map $f : \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathbb{T}^2 \setminus \{0\}$ is of the form*

$$f(q_1, q_2, \dots, q_n) = \mathbf{m}(q_i - q_j) \text{ with some } i \neq j,$$

where $\mathbf{m} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is either the identity or a complex multiplication.⁷

NOTATION 3.2. We denote by \mathfrak{M} the finite cyclic group of automorphisms of \mathbb{T}^2 consisting of \pm identity and all complex multiplications in \mathbb{T}^2 (if they exist). \mathfrak{M} is isomorphic either to \mathbb{Z}_2 or to \mathbb{Z}_4 or to \mathbb{Z}_6 . Let \mathfrak{M}_+ consists of all $\mathbf{m} \in \mathfrak{M}$ with $0 \leq \text{Arg } \mathbf{m} < \pi$ (i.e. \mathfrak{M}_+ contains 1, 2 or 3 elements).

To prove the theorem we need some preparation. The proof of the following well-known statement is due to H. Huber [19], §6, Satz 2; it may also be found in Sh. Kobayashi's book [20], Chapter VI, Sec. 2, remarks after Corollary 2.6.

CLAIM A. *Let M and N be compact Riemann surfaces and $A \subset M$ and $B \subset N$ be finite sets. Suppose that the domain $N \setminus B$ is hyperbolic, meaning that its universal covering is isomorphic to the unit disc. Then any holomorphic map $M \setminus A \rightarrow N \setminus B$ extends to a holomorphic map $M \rightarrow N$.*

The following facts are also very well known.

CLAIM B. *Any non-constant holomorphic self-map $\mu : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is regular, has no critical points and hence is an unbranched covering of some finite degree k . If $\mu(0) = 0$ then μ is also a regular endomorphism of the compact algebraic group \mathbb{T}^2 and its kernel $\text{Ker } \mu$ is a subgroup of order k . Every holomorphic self-map of \mathbb{T}^2 is of the form $q \mapsto \sum_{\mathbf{m} \in \mathfrak{M}} k_{\mathbf{m}} \mathbf{m}(q - a)$, and any automorphism is of the form $q \mapsto \mathbf{m}(q - a)$,*

where $\mathbf{m} \in \mathfrak{M}$, $k_{\mathbf{m}} \in \mathbb{Z}$, and $a \in \mathbb{T}^2$.

Explanation. The graph $G_\mu = \{(p, q) \in \mathbb{T}^2 \times \mathbb{T}^2 \mid q = \mu(p)\}$ of μ is an analytic subset of the projective variety $\mathbb{T}^2 \times \mathbb{T}^2$. By general Chow's theorem (see [21] or [22], Chapter V, Theorem D7), G_μ is a (non-singular) projective variety. Of course, the projections $\alpha : G_\mu \ni (p, q) \mapsto p \in \mathbb{T}^2$ and $\beta : G_\mu \ni (p, q) \mapsto q \in \mathbb{T}^2$ are regular. Hence, the inverse map $\alpha^{-1} : \mathbb{T}^2 \ni p \mapsto (p, \mu(p)) \in G_\mu$ is regular, and the composition $\mu = \beta \circ \alpha^{-1}$ is regular, as well.

To see that μ has no critical points, suppose to the contrary that the set of all critical values C contains $m \geq 1$ points. The restriction of μ to the complement of $\mu^{-1}(C)$ defines an unbranched covering $\mathbb{T}^2 \setminus \mu^{-1}(C) \rightarrow \mathbb{T}^2 \setminus C$ of some finite

⁷Of course, the latter may happen only if our torus admits such a multiplication (by a complex multiplication we mean an automorphism of the torus \mathbb{T}^2 induced by multiplication on the complex line by a complex number $\neq 1, -1$).

degree k . The Riemann–Hurwitz formula for Euler characteristics states that $\chi(\mathbb{T}^2 \setminus \mu^{-1}(C)) = k\chi(\mathbb{T}^2 \setminus C)$, i.e. $\#(\mu^{-1}(C)) = km$. Since $\#(\mu^{-1}(c)) \leq k-1$ for all $c \in C$, we have $km = \#(\mu^{-1}(C)) \leq (k-1)m$, which is impossible.

Assuming that $\mu(0) = 0$, define a map $\nu: \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ by

$$\nu(p, q) = \mu(p + q) - \mu(p) - \mu(q).$$

Any loop in $\mathbb{T}^2 \times \mathbb{T}^2$ can be deformed to the subset $(\mathbb{T}^2 \times \{0\}) \cup (\{0\} \times \mathbb{T}^2)$, where $\nu = 0$. Thus, ν induces the trivial homomorphism of the fundamental groups and lifts to a holomorphic map $\tilde{\nu}: \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{C}$, which, by the maximum principle, must be constant. It follows that $\nu = \text{const} = 0$ and μ is a holomorphic group endomorphism of \mathbb{T}^2 . (Another argument may be found in [23], Chapter 3, Section 3.1.)

The general form of automorphisms and endomorphisms of \mathbb{T}^2 may be found in [24], Chapter V, Section V.4.7, and [25], Chapter 1, Section 5, respectively. \square

LEMMA 3.3. *For any $a, b \in \mathbb{T}^2$ every non-constant holomorphic map $\lambda: \mathbb{T}^2 \setminus \{a\} \rightarrow \mathbb{T}^2 \setminus \{b\}$ extends to a biregular automorphism of \mathbb{T}^2 sending a to b .*

Proof. By Claims A and B, λ extends to an unbranched holomorphic covering map $\tilde{\lambda}: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ of some finite degree k . Clearly $\tilde{\lambda}^{-1}(a) = \{b\}$; hence, $k = 1$ and $\tilde{\lambda}$ is an automorphism. \square

DEFINITION 3.4. A configuration $a = (a_1, \dots, a_m) \in \mathcal{E}^m(\mathbb{T}^2)$ is said to be *exceptional* if there exist $i \neq j$ and a non-constant holomorphic self-map $\lambda: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $\lambda(a_i) = \lambda(a_j)$ and $\lambda^{-1}(\lambda(a_i)) \subseteq \{a_1, \dots, a_m\}$.

LEMMA 3.5. *a) The set A of all exceptional configurations $a = (a_1, \dots, a_m) \in \mathcal{E}^m(\mathbb{T}^2)$ is contained in a subvariety $M \subset \mathcal{E}^m(\mathbb{T}^2)$ of codimension 1.*

b) For any non-exceptional configuration $(a_1, \dots, a_m) \in \mathcal{E}^m(\mathbb{T}^2)$, every non-constant holomorphic map $\lambda: \mathbb{T}^2 \setminus \{a_1, \dots, a_m\} \rightarrow \mathbb{T}^2 \setminus \{0\}$ extends to a biregular automorphism of \mathbb{T}^2 , sending a certain a_i to 0.

Proof. a) Let N denote the union of all finite subgroups of order $\leq m$ in \mathbb{T}^2 ; this set is finite. Set $M = \{(a_1, \dots, a_m) \in \mathcal{E}^m(\mathbb{T}^2) \mid a_j - a_i \in N \text{ for some } i \neq j\}$; then M is a subvariety in $\mathcal{E}^m(\mathbb{T}^2)$ of codimension 1. We shall show that $A \subseteq M$.

Let $a = (a_1, \dots, a_m) \in A$ and let i, j , and λ be as in Definition 3.4. Set $\mu(t) = \lambda(t + a_i) - \lambda(a_i)$, $t \in \mathbb{T}^2$. Then $\mu(0) = 0$ and, by Claim B, μ is a group homomorphism with finite kernel $\# \text{Ker } \mu$. If $t \in \text{Ker } \mu$, then $\lambda(t + a_i) = \lambda(a_i)$, $t + a_i \in \lambda^{-1}(\lambda(a_i)) \subseteq \{a_1, \dots, a_m\}$ and $t \in \{a_1 - a_i, \dots, 0, \dots, a_m - a_i\}$; that is, $\text{Ker } \mu \subseteq \{a_1 - a_i, \dots, 0, \dots, a_m - a_i\}$. In particular, $\# \text{Ker } \mu \leq m$ and hence $\text{Ker } \mu \subseteq N$. Since $\mu(a_j - a_i) = 0$, we have $a_j - a_i \in N$ and $a \in M$.

b) Let $a = (a_1, \dots, a_m) \notin A$. By Claims A and B, any non-constant holomorphic map $\lambda: \mathbb{T}^2 \setminus \{a_1, \dots, a_m\} \rightarrow \mathbb{T}^2 \setminus \{0\}$ extends to a finite unbranched holomorphic covering map $\tilde{\lambda}: \mathbb{T}^2 \rightarrow \mathbb{T}^2$. Clearly $\tilde{\lambda}^{-1}(0) \subseteq \{a_1, \dots, a_m\}$; in particular, $\tilde{\lambda}(a_i) = 0$ for a certain i . Since $a \notin A$, we have $\tilde{\lambda}(a_j) \neq 0$ for all $j \neq i$; this means that $\tilde{\lambda}^{-1}(0) = \{a_i\}$ and $\deg \tilde{\lambda} = 1$. \square

Proof of Theorem 3.1 First, we notice that any holomorphic map $\mathbb{T}^2 \rightarrow \mathbb{T}^2 \setminus \{0\}$ is constant, since it lifts to a holomorphic map $\mathbb{C} \rightarrow \mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ of the universal coverings, which is constant due to Liouville's theorem.

The proof of the theorem is by induction on n , starting with $n = 2$. For $a \in \mathbb{T}^2$, denote by $\lambda_a = \lambda(\cdot, a)$ the restriction of λ to the fibre $p^{-1}(a) = \mathbb{T}^2 \setminus \{a\}$ of the projection $p: \mathcal{E}^2(\mathbb{T}^2) \ni (q_1, q_2) \mapsto q_2 \in \mathbb{T}^2$.

The set $S = \{a \in \mathbb{T}^2 \mid \lambda_a \text{ is a constant map}\}$ is finite. Because, otherwise, by the uniqueness theorem, $\lambda_a = \text{const}$ for all $a \in \mathbb{T}^2$, $\lambda = \lambda(q_1, q_2)$ does not depend on q_1 and may be considered as a holomorphic map $\mathbb{T}^2 \rightarrow \mathbb{T}^2 \setminus \{0\}$, which must be constant; however, this implies $\lambda = \text{const}$, contradicting our assumption.

By Lemma 3.3, for any $a \notin S$ the map $\lambda_a: \mathbb{T}^2 \setminus \{a\} \rightarrow \mathbb{T}^2 \setminus \{0\}$ extends to a biholomorphic automorphism of \mathbb{T}^2 sending a to 0. This means that λ_a is of the form $\lambda_a(t) = \mathbf{m}(t - a)$, with some $\mathbf{m} = \mathbf{m}_a \in \mathfrak{M}$. Thus, for all (q_1, q_2) in the connected, everywhere dense set $\mathcal{E}^2(\mathbb{T}^2) \setminus p^{-1}(S)$ we have

$$\lambda(q_1, q_2) = \mathbf{m}(q_1 - q_2) \quad (3.1)$$

with a certain $\mathbf{m} = \mathbf{m}_{q_2} \in \mathfrak{M}$. Since \mathfrak{M} is finite, the element $\mathbf{m} = \mathbf{m}_{q_2} \in \mathfrak{M}$ on the right hand side of (3.1) cannot depend on q_2 and the latter formula holds true for the whole of $\mathcal{E}^2(\mathbb{T}^2)$, which completes the proof for $n = 2$.

Assume that the theorem is already proved for some $n = m - 1 \geq 2$. For $a = (a_2, \dots, a_m) \in \mathcal{E}^{m-1}(\mathbb{T}^2)$, denote by $\lambda_a = \lambda(\cdot, a_2, \dots, a_m)$ the restriction of λ to the fibre $p^{-1}(a) = \mathbb{T}^2 \setminus \{a_2, \dots, a_m\}$ of the projection $p: \mathcal{E}^m(\mathbb{T}^2) \ni (q_1, q_2, \dots, q_m) \mapsto (q_2, \dots, q_m) \in \mathcal{E}^{m-1}(\mathbb{T}^2)$.

It is clear that $S \stackrel{\text{def}}{=} \{a \in \mathcal{E}^{m-1}(\mathbb{T}^2) \mid \lambda_a \text{ is a constant map}\}$ is an analytic subset of $\mathcal{E}^{m-1}(\mathbb{T}^2)$, and either (i) $S = \mathcal{E}^{m-1}(\mathbb{T}^2)$ or (ii) $\dim_{\mathbb{C}} S \leq m - 2$.

In case (i), $\lambda = \lambda(q_1, \dots, q_m)$ does not depend on q_1 and may be considered as a holomorphic map $\mathcal{E}^{m-1}(\mathbb{T}^2) \rightarrow \mathbb{T}^2 \setminus \{0\}$; by the induction hypothesis, λ is of the desired form.

Let us consider case (ii). By Lemma 3.5(a), the set A of all exceptional configurations is contained in a subvariety $M \subset \mathcal{E}^{m-1}(\mathbb{T}^2)$ of dimension $m - 2$. Let $a \in \mathcal{E}^{m-1}(\mathbb{T}^2) \setminus (S \cup M)$. Then $\lambda_a: \mathbb{T}^2 \setminus \{a_2, \dots, a_m\} \rightarrow \mathbb{T}^2 \setminus \{0\}$ is a non-constant map. By Lemma 3.5(b), λ_a extends to a biholomorphic map $\tilde{\lambda}_a: \mathbb{T}^2 \rightarrow \mathbb{T}^2$. Therefore, $\tilde{\lambda}_a(t) = \mathbf{m}(t - a_i)$, with some $\mathbf{m} = \mathbf{m}_a \in \mathfrak{M}$ and $i = i_a$. Thus, for all (q_1, \dots, q_m) in the connected, everywhere dense set $\mathcal{E}^m(\mathbb{T}^2) \setminus p^{-1}(S \cup M)$ we have

$$\lambda(q_1, \dots, q_m) = \mathbf{m}(q_1 - q_i) \quad (3.2)$$

with certain $\mathbf{m} = \mathbf{m}_q \in \mathfrak{M}$ and $i = i_q$. Since \mathfrak{M} is finite, \mathbf{m} and i do not depend on q , and (3.2) holds true on the whole of $\mathcal{E}^m(\mathbb{T}^2)$, which completes the step of induction thus proving the theorem. \square

DEFINITION 3.6. For any $\mathbf{m} \in \mathfrak{M}_+$ and $i \neq j \in \{1, \dots, n\}$, the map

$$\begin{aligned} e_{\mathbf{m};i,j}: \mathcal{E}^n(\mathbb{T}^2) &\rightarrow \mathbb{T}^2 \setminus \{0\}, \\ e_{\mathbf{m};i,j}(q) &= \mathbf{m}(q_i - q_j) \text{ for } q = (q_1, \dots, q_n) \in \mathcal{E}^n(\mathbb{T}^2), \end{aligned} \quad (3.3)$$

is called a *difference*. For a difference $\mu = e_{\mathbf{m};i,j}$, the unordered pair of variables $\{q_i, q_j\}$ is called the *support* of μ and the automorphism $\mathbf{m} \in \mathfrak{M}_+$ is called the *marker* of μ . We denote them by $\text{supp } \mu$ and \mathbf{m}_μ respectively. It follows from Theorem 3.1 that any non-constant holomorphic map $\mu: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathbb{T}^2 \setminus \{0\}$ admits a unique representation in the form of a difference, that is, $\mu = e_{\mathbf{m};i,j}$ for some uniquely defined $\mathbf{m} \in \mathfrak{M}_+$ and $i, j \in \{1, \dots, n\}$.

3.3. A simplicial structure on the set of differences. The set of all non-constant holomorphic maps of a complex space to a punctured torus carries a natural simplicial structure⁸.

DEFINITION 3.7. For a connected complex space Y , let $L(Y)$ denote the set of all non-constant holomorphic maps $\mu: Y \rightarrow \mathbb{T}^2 \setminus \{0\}$. For $\mu, \nu \in L(Y)$, we say that ν is a *proper reminder* of μ and write $\nu \mid \mu$ if $\mu - \nu \in L(Y)$. This relation is symmetric, i.e. $\nu \mid \mu$ is equivalent to $\mu \mid \nu$.

We define the graph $\Gamma(Y)$ with the set of vertices $L(Y)$ as follows: $\{\mu, \nu\}$ is an *edge* connecting μ and ν whenever $\mu \mid \nu$. We denote by $L_\Delta(Y)$ the *flag complex* of the graph $\Gamma(Y)$. This means that a subset $\Delta^m = \{\mu_0, \dots, \mu_m\} \subseteq L(Y)$ is said to be an *m-simplex* in $L_\Delta(Y)$ if $\{\mu_i, \mu_j\}$ is an edge in $\Gamma(Y)$ for all $i \neq j$, i.e. $\mu_i \mid \mu_j$ for all $i \neq j$.

LEMMA 3.8. *Let $f: Z \rightarrow Y$ be a holomorphic map of connected complex spaces. Suppose that for each $\lambda \in L(Y)$ the composition*

$$f^*(\lambda) \stackrel{\text{def}}{=} \lambda \circ f: Z \xrightarrow{f} Y \xrightarrow{\lambda} \mathbb{T}^2 \setminus \{0\} \quad (3.4)$$

is non-constant. Then

$$f^*: L(Y) \ni \lambda \mapsto \lambda \circ f \in L(Z) \quad (3.5)$$

is a simplicial map whose restriction to any simplex $\Delta \subseteq L(Y)$ is injective. In particular, f^ preserves dimension of simplices.*

Proof. For any $\lambda \in L(Y)$, $f^*(\lambda)$ is a non-constant holomorphic map to $\mathbb{T}^2 \setminus \{0\}$; hence $f^*(\lambda) \in L(Z)$. If $\mu, \nu \in L(Y)$ and $\mu \mid \nu$, then $\lambda = \mu - \nu \in L(Y)$ and $f^*(\mu) - f^*(\nu) = f^*(\mu - \nu) = f^*(\lambda) \in L(Z)$; consequently, $f^*(\mu) \mid f^*(\nu)$. This implies that the map f^* is simplicial and injective on any simplex. \square

REMARK 3.9. Condition $f^*(\lambda) \neq \text{const}$ for all $\lambda \in L(Y)$ is certainly fulfilled for any regular dominant map $f: Y \rightarrow Z$ of non-singular irreducible algebraic varieties.

We need to study some properties of complexes $L_\Delta(Y)$ for the case when Y is an ordered configuration space of a torus. Notice that according to Theorem 3.1, the set $L(\mathcal{E}^n(\mathbb{T}^2))$ coincides with the set of all differences on $\mathcal{E}^n(\mathbb{T}^2)$.

LEMMA 3.10. *Let $\{\mu_0, \dots, \mu_s\} \in L_\Delta(\mathcal{E}^n(\mathbb{T}^2))$ be an s -simplex. Then $\mathbf{m}_{\mu_i} = \mathbf{m}_{\mu_j}$, $\#(\text{supp } \mu_i \cap \text{supp } \mu_j) = 1$ for all $i \neq j$, and $\#(\text{supp } \mu_0 \cap \dots \cap \text{supp } \mu_s) = 1$.*

Proof. Let $i \neq j$ and let $\mu_i = \mathbf{m}_i(q_{i'} - q_{i''})$ and $\mu_j = \mathbf{m}_j(q_{j'} - q_{j''})$. Since $\mu_i \mid \mu_j$, we must have $\mu_i - \mu_j = \mathbf{m}(q_{k'} - q_{k''})$ for some $\mathbf{m} \in \mathfrak{M}_+$ and $k' \neq k''$. Thus, $\mathbf{m}_i(q_{i'} - q_{i''}) - \mathbf{m}_j(q_{j'} - q_{j''}) = \mathbf{m}(q_{k'} - q_{k''})$. The latter relation can be fulfilled only if either $\mathbf{m}_i q_{i'} - \mathbf{m}_j q_{j'} = 0$ or $\mathbf{m}_i q_{i''} - \mathbf{m}_j q_{j''} = 0$. This implies $\mathbf{m}_i = \mathbf{m}_j$ and we have

$$\text{either } i' = j' \quad \text{or} \quad i'' = j''. \quad (3.6)$$

If $s = 1$ we have finished the proof. If $s > 2$, then the property $\#(\text{supp } \mu_i \cap \text{supp } \mu_j) = 1$ implies immediately that $\#(\text{supp } \mu_0 \cap \dots \cap \text{supp } \mu_s) = 1$. For $s = 2$ we have

$$\mu_0 = \mathbf{m}(q_{i'} - q_{i''}), \quad \mu_1 = \mathbf{m}(q_{j'} - q_{j''}), \quad \mu_2 = \mathbf{m}(q_{k'} - q_{k''}).$$

⁸Compare to [3]

Since $\mu_0 \mid \mu_1$, $\mu_1 \mid \mu_2$ and $\mu_2 \mid \mu_0$, we obtain that

$$\#(\text{supp } \mu_0 \cap \text{supp } \mu_1) = \#(\text{supp } \mu_1 \cap \text{supp } \mu_2) = \#(\text{supp } \mu_2 \cap \text{supp } \mu_0) = 1.$$

Let $N = \#(\text{supp } \mu_0 \cap \text{supp } \mu_1 \cap \text{supp } \mu_2)$. Clearly $N \leq 1$; let us show that $N \neq 0$. Suppose to the contrary that $N = 0$. Relations (3.6) apply to μ_0 and μ_1 , and without loss of generality we can assume that $i' = j'$. For μ_1 and μ_2 the same relations tell us that either $j' = k'$ or $j'' = k''$; since $N = 0$, the first case is impossible and we are left with $j'' = k''$. Finally, we apply (3.6) to μ_0 and μ_2 and see that either $i' = k'$ or $i'' = k''$, which leads to a contradiction and completes the proof. \square

The $\mathbf{S}(n)$ action in $\mathcal{E}^n(\mathbb{T}^2)$ induces an $\mathbf{S}(n)$ action in $L(\mathcal{E}^n(\mathbb{T}^2))$ defined by $(\sigma\lambda)(q) = \lambda(\sigma^{-1}q)$. The latter action, in turn, induces a simplicial $\mathbf{S}(n)$ action in the complex $L_\Delta(\mathcal{E}^n(\mathbb{T}^2))$ which preserves dimension of simplices; our nearest goal is to describe the orbits of this action.

DEFINITION 3.11. We define the following *normal forms* of simplices of dimension $s > 0$: $\Delta_{\mathbf{m}}^s = \{e_{\mathbf{m};1,2}, \dots, e_{\mathbf{m};1,s+2}\}$, $\nabla_{\mathbf{m}}^s = \{e_{\mathbf{m};2,1}, \dots, e_{\mathbf{m};s+2,1}\}$, where $\mathbf{m} \in \mathfrak{M}_+$; these simplices are called *normal*.

LEMMA 3.12. *For $s > 0$, there are exactly $\#\mathfrak{M}$ orbits of the $\mathbf{S}(n)$ action on the set of all s -simplices.⁹ Every orbit contains exactly one normal simplex.*

Proof. Since $e_{\mathbf{m};a,b} \nmid e_{\mathbf{m};b,c}$, Lemma 3.10 implies that any s -simplex $\Delta \in L_\Delta(\mathcal{C}^n(\mathbb{T}^2))$ is either of the form $\{e_{\mathbf{m};a,b_0}, \dots, e_{\mathbf{m};a,b_s}\}$ or of the form $\{e_{\mathbf{m};b_0,a}, \dots, e_{\mathbf{m};b_s,a}\}$ with some $\mathbf{m} \in \mathfrak{M}_+$ and distinct a, b_0, \dots, b_s . An appropriate permutation $\sigma \in \mathbf{S}(n)$ carries Δ to a normal form. \square

COROLLARY 3.13. $\dim L_\Delta(\mathcal{E}^n(\mathbb{T}^2)) = n - 2$.

3.4. Regular mappings $\mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathbb{T}^2$. The following lemma will be used in Section 4.

LEMMA 3.14. *Let $\lambda : (\mathbb{T}^2)^n \rightarrow \mathbb{T}^2$ be a rational map. Then it is regular and there are a holomorphic self-maps μ_1, \dots, μ_n of \mathbb{T}^2 such that $\lambda(q_1, \dots, q_n) = \sum_{i=1}^n \mu_i(q_i)$. In particular,*

$$\lambda(q_1, \dots, q_n) = \sum_{i=1}^n \sum_{\mathbf{m} \in \mathfrak{M}_+} k_{i,\mathbf{m}} \mathbf{m} q_i + c \quad \text{for all } (q_1, \dots, q_n) \in \mathcal{E}^n(\mathbb{T}^2),$$

where $k_{i,\mathbf{m}} \in \mathbb{Z}$ and $c \in \mathbb{T}^2$.

Proof. The proof is by induction on n . For $n = 1$, λ is a rational map of smooth projective curves. It extends to a regular map (see, for instance, [26], Chapter II, Sec. 3.1, Corollary 1). By Claim B, the regular self-map λ of \mathbb{T}^2 is of the desired form.

Assume that the theorem has already been proved for some $n = m - 1 \geq 1$. There is a subset $\Sigma \subset (\mathbb{T}^2)^m$ of codimension 1 such that λ is regular on $(\mathbb{T}^2)^m \setminus \Sigma$. Let $(t_0, z_0) \in (\mathbb{T}^2 \times (\mathbb{T}^2)^{m-1}) \setminus \Sigma$ and D be a small neighbourhood of z_0 in $(\mathbb{T}^2)^{m-1}$. Without loss of generality, we may assume that $t_0 = 0$ and $(0, z) \notin \Sigma$ for all $z \in D$.

⁹The number of $\mathbf{S}(n)$ orbits on the set of all 0-simplices is $\#\mathfrak{M}/2$.

For $(t, z) \in (\mathbb{T}^2 \times D) \setminus \Sigma$, set $\mu(t, z) = \lambda(t, z) - \lambda(0, z)$ and $\nu(t, z) = \mu(t, z) - \mu(t, z_0)$. For any $z \in D$, we have $\nu(0, z) = 0$ and hence the map $t \mapsto \nu(t, z)$ extends to a holomorphic endomorphism ν_z of the torus \mathbb{T}^2 ; moreover, the endomorphism ν_{z_0} carries the whole torus \mathbb{T}^2 to the zero point $0 \in \mathbb{T}^2$. One can find a neighbourhood $D' \Subset D$ of z_0 and a compact subset $K \subset \mathbb{T}^2 \times D$ such that for all $z \in D'$ the intersection $K \cap (\mathbb{T}^2 \times \{z\})$ is a union of two loops that do not meet Σ and generate the whole fundamental group of the torus $\mathbb{T}^2 \times \{z\}$. Moreover, since $\nu(\mathbb{T}^2 \times \{z_0\}) = 0$, we may also assume that $\nu(K)$ is contained in a small contractible neighbourhood of the zero point in \mathbb{T}^2 . It follows that for any $z \in D'$ the endomorphism ν_z is contractible and hence trivial. Thus, for such z we have $\mu(t, z) - \mu(t, z_0) \equiv 0$ and $\lambda(t, z) \equiv \lambda(0, z) + \lambda(t, z_0) - \lambda(0, z_0)$. By the uniqueness theorem, the latter identity holds true on the whole of $(\mathbb{T}^2 \times (\mathbb{T}^2)^{m-1}) \setminus \Sigma$; the inductive hypothesis applies to $\lambda(0, z)$ and $\lambda(t, z_0)$, which leads to the desired representation of $\lambda(t, z)$ and completes the proof. \square

4. HOLOMORPHIC MAPPINGS OF CONFIGURATION SPACES

The main goal of this section is to prove the classification theorem 1.1 for holomorphic endomorphisms of torus configuration spaces. We are proceeding according to the plan exposed in Introduction.

4.1. Strictly equivariant mappings. A lifting f of a continuous self-map F of $\mathcal{C}^n(X)$ to the covering $\mathcal{E}^n(X)$ is equivariant, meaning that there is an endomorphism α of the symmetric group $\mathbf{S}(n)$ such that $f(\sigma q) = \alpha(\sigma)f(q)$ for all $q \in \mathcal{E}^n(X)$ and $\sigma \in \mathbf{S}(n)$. We will see that a lifting of a *non-abelian* F has a stronger property defined below.

DEFINITION 4.1. A continuous map $f: \mathcal{E}^n(X) \rightarrow \mathcal{E}^n(X)$ is said to be *strictly equivariant* if there exists an automorphism α of the group $\mathbf{S}(n)$ such that

$$f(\sigma q) = \alpha(\sigma)f(q) \text{ for all } q \in \mathcal{E}^n(X) \text{ and } \sigma \in \mathbf{S}(n). \quad (4.1)$$

THEOREM 4.2. a) For $n > 4$ any non-abelian continuous map $F: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ admits a continuous lifting $f: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathcal{E}^n(\mathbb{T}^2)$ which fits in the diagram (1.1).

b) For $n > 4$ any continuous lifting $f: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathcal{E}^n(\mathbb{T}^2)$ of a non-abelian continuous map $F: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ is strictly equivariant.

Proof. In view of the covering mapping theorem, (a) follows from Theorem 1.4. Let us prove (b). The diagram (1.1) for f and F leads to the algebraic commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\mathcal{E}^n(\mathbb{T}^2), Q^\circ) & \xrightarrow{p_*} & \pi_1(\mathcal{C}^n(\mathbb{T}^2), Q^\circ) & \xrightarrow{\delta} & \mathbf{S}(n) \longrightarrow 1 \\ & & \downarrow f_* & & \downarrow F_* & & \downarrow \alpha \\ 1 & \longrightarrow & \pi_1(\mathcal{E}^n(\mathbb{T}^2), f(Q^\circ)) & \xrightarrow{p_*} & \pi_1(\mathcal{C}^n(\mathbb{T}^2), f(Q^\circ)) & \xrightarrow{\delta} & \mathbf{S}(n) \longrightarrow 1, \end{array} \quad (4.2)$$

which relates the final segments of the exact homotopy sequences of the coverings p in the left and right columns of (1.1). The condition (4.1) holds true with the endomorphism α that appears in (4.2), and we have only to show that this α is an automorphism whenever F^* is non-abelian and $n > 4$.

Suppose to the contrary that α is not an automorphism; then its image is a non-trivial quotient of $\mathbf{S}(n)$, which must be abelian since $n > 4$. In view of (4.2), the composition $\delta \circ F^* = \alpha \circ \delta$ is abelian and, by Lemma 2.3, F^* itself is abelian, which contradicts our assumption. \square

The following lemma shows that the assertion of Lemma 3.8 holds true for every strictly equivariant map.

LEMMA 4.3. *Let $n > 2$ and $f = (f_1, \dots, f_n): \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathcal{E}^n(\mathbb{T}^2)$ be a strictly equivariant holomorphic map. Then*

$$f^*: L(\mathcal{E}^n(\mathbb{T}^2)) \ni \lambda \mapsto \lambda \circ f \in L(\mathcal{E}^n(\mathbb{T}^2))$$

is a well-defined simplicial map whose restriction to each simplex $\Delta \subseteq L(\mathcal{E}^n(\mathbb{T}^2))$ is injective. Consequently, f^ preserves dimension of simplices.*

Proof. In view of Lemma 3.8, we have only to prove that $\mu \circ f \neq \text{const}$ for any $\mu \in L(\mathcal{E}^n(\mathbb{T}^2))$. Suppose to the contrary that $\mu \circ f = c \in \mathbb{T}^2$. Then $(\mu \circ f)(\sigma q) \equiv c$ for all $\sigma \in \mathbf{S}(n)$. Since f is strictly equivariant, there is $\alpha \in \text{Aut } \mathbf{S}(n)$ such that $f(\sigma q) = \alpha(\sigma)f(q)$ for all $\sigma \in \mathbf{S}(n)$ and $q \in \mathcal{E}^n(X)$, so that $c \equiv \mu(f(\sigma q)) = \mu(\alpha(\sigma)f(q))$.

By Theorem 3.1, $\mu = \mathbf{m}(q_i - q_j)$ for some distinct i, j and $\mathbf{m} \in \mathfrak{M}$; hence $c \equiv (\mu \circ f)(q) = \mathbf{m}(f_i(q) - f_j(q))$. Since α is an automorphism and $n > 2$, there is $\sigma \in \mathbf{S}(n)$ such that $\alpha(\sigma^{-1})(i) = i$ and $\alpha(\sigma^{-1})(j) = k \neq j$; thus, for all q we have

$$\begin{aligned} \mathbf{m}(f_i(q) - f_j(q)) &= c = (\mu \circ f)(\sigma q) = \mu(\alpha(\sigma)f(q)) \\ &= \mathbf{m}(f_{\alpha(\sigma^{-1})(i)}(q) - f_{\alpha(\sigma^{-1})(j)}(q)) = \mathbf{m}(f_i(q) - f_k(q)), \end{aligned}$$

which is impossible. \square

4.2. **Proof of Theorem 1.1.** We shall prove two theorems, which together imply Theorem 1.1.

THEOREM 4.4. *For $n > 4$, every holomorphic non-abelian self-map F of $\mathcal{C}^n(\mathbb{T}^2)$ is tame.*

Proof. By Theorems 1.4 and 4.2, the map F lifts to a strictly equivariant holomorphic map f that fits to the commutative diagram

$$\begin{array}{ccc} \mathcal{E}^n(\mathbb{T}^2) & \xrightarrow{f} & \mathcal{E}^n(\mathbb{T}^2) \\ p \downarrow & & \downarrow p \\ \mathcal{C}^n(\mathbb{T}^2) & \xrightarrow{F} & \mathcal{C}^n(\mathbb{T}^2). \end{array}$$

Let α be the automorphism of $\mathbf{S}(n)$ corresponding to f (see Definition 4.1).

By Lemma 4.3, f^* is a dimension preserving simplicial self-map of $L_\Delta(\mathcal{E}^n(\mathbb{T}^2))$. Let $\Delta_1 = \{q_1 - q_2, \dots, q_1 - q_n\}$ and $\Delta = f^*(\Delta_1)$. By Lemma 3.12, there exists a permutation σ that brings Δ to its normal form; without loss of generality, we may assume that this normal form is $\nabla_{\mathbf{m}} = \{\mathbf{m}(q_2 - q_1), \dots, \mathbf{m}(q_n - q_1)\}$, where $\mathbf{m} \in \mathfrak{M}_+$. Set $\tilde{f} = f \circ \sigma$; then

$$\tilde{f}_j = \tilde{f}_1 + \mathbf{m}(q_1 - q_j), \quad j = 1, \dots, n. \quad (4.3)$$

Define the holomorphic map $\tau: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \text{Aut}(\mathbb{T}^2)$ by the condition

$$\tau(q)(z) = \tau(q_1, \dots, q_n)(z) = -\mathbf{m}z + (\tilde{f}_1(q) + \mathbf{m}q_1),$$

where $q = (q_1, \dots, q_n) \in \mathcal{E}^n(\mathbb{T}^2)$ and $z \in \mathbb{T}^2$. Equations (4.3) imply that $\tau(q)q_j = f_j(\sigma q)$ for all $j = 1, \dots, n$ and $q = (q_1, \dots, q_n) \in \mathcal{E}^n(\mathbb{T}^2)$; thereby $\tau(q)q = f(\sigma q) = \alpha(\sigma)f(q)$, or, which is the same, $f(q) = \alpha(\sigma^{-1})\tau(q)q$ for all $q \in \mathcal{E}^n(\mathbb{T}^2)$. To

complete the proof, we must check that τ is $\mathbf{S}(n)$ -invariant. For every $s \in \mathbf{S}(n)$ and all $q \in \mathcal{E}^n(\mathbb{T}^2)$ we have

$$\begin{aligned} s\tau(sq)q &= \tau(sq)sq = f(\sigma sq) = \alpha(\sigma s)f(q) \\ &= \alpha(\sigma s)f(\sigma\sigma^{-1}q) = \alpha(\sigma s)\tau(\sigma^{-1}q)\sigma^{-1}q = \alpha(\sigma s)\sigma^{-1}\tau(\sigma^{-1}q)q, \end{aligned}$$

which can be rewritten as

$$[(\tau(sq))^{-1} \cdot \tau(\sigma^{-1}q)]q = \sigma\alpha(s^{-1}\sigma^{-1})sq, \quad (4.4)$$

where $(\tau(sq))^{-1} \cdot \tau(\sigma^{-1}q) \in \text{Aut}(\mathbb{T}^2)$ is the product in group $\text{Aut}(\mathbb{T}^2)$. Let us notice that for $n > 1$ there is a non-empty Zariski open subset $A \subset \mathcal{E}^n(\mathbb{T}^2)$ such that if $\theta q = \rho q$ for some $q \in A$, $\theta \in \text{Aut}(\mathbb{T}^2)$ and $\rho \in \mathbf{S}(n)$, then $\theta = \text{id}$ and $\rho = 1$. Therefore, equation (4.4) implies $\tau(sq) = \tau(\sigma^{-1}q)$ and $\sigma\alpha(s^{-1}\sigma^{-1})s = 1$ for all $q \in A$ and all $s \in \mathbf{S}(n)$. Since τ is continuous, the first of these relations holds true for all $q \in \mathcal{E}^n(\mathbb{T}^2)$ and all $s \in \mathbf{S}(n)$, which shows that $\tau: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \text{Aut}(\mathbb{T}^2)$ is invariant¹⁰. \square

REMARK 4.5. Let n be either 3 or 4. The statement of Theorem 4.4 still holds true if we assume that the map F is an automorphism. The only changes we need to make in the proof are as follows: instead of our Theorem 4.2, we have to use Theorem 2 from [13], which states that the pure braid group is a characteristic subgroup of the torus braid group; moreover, instead of Lemma 4.3, we should use Remark 3.9 from Section 3.3. The rest of the proof is the same.

REMARK 4.6. a) Let $n \geq 2$ and let $F: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ be a tame map. Then a morphism $T: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \text{Aut } \mathbb{T}^2$ in the 'tame representation' $F = F_T$ of F is uniquely determined by F . Indeed, if $F_T = F_{T'}$ for two morphisms T, T' , then $T(Q)Q = T'(Q)Q$ and $(*) [T(Q)]^{-1}T'(Q)Q = Q$ for all $Q \in \mathcal{C}^n(\mathbb{T}^2)$. Furthermore, a torus automorphism is uniquely determined by its values at a generic pair of distinct points; since $n \geq 2$, the identity $(*)$ shows that $[T(Q)]^{-1}T'(Q) = \text{id}$ for any generic point $Q \in \mathcal{C}^n(\mathbb{T}^2)$ and hence $T(Q) = T'(Q)$ everywhere.

b) In view of Theorem 4.4, (a) shows that for $n > 4$ any holomorphic non-abelian map $F: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ admits a unique tame representation $F = F_T$ and the morphism T is regular whenever F is so. Remark 4.5 shows that the uniqueness (and regularity) of T still hold true whenever n is 3 or 4 and F is a (biregular) automorphism.

DEFINITION 4.7. The map

$$\mathcal{C}^n(\mathbb{T}^2) \ni q = \{q_1, \dots, q_n\} \mapsto s(q) = (q_1 + \dots + q_n) \in \mathbb{T}^2$$

is a locally trivial holomorphic fibration whose fibre $M_0 = s^{-1}(0)$ is an algebraic variety. We refer to the variety M_0 as the *reduced configuration space*. The presentation of the fundamental group $\pi_1(M_0)$ found by O. Zariski [14] shows that $H_1(M_0, \mathbb{Z}) = \mathbb{Z}_{2n}$; Zariski called the group $\pi_1(M_0)$ the *invariant subgroup of the group of motion classes of an elliptic Riemann surface*.

Let $\gamma: \mathbb{C} \rightarrow \mathbb{T}^2$ be the universal covering; then the mappings

$$\begin{aligned} M_0 \times \mathbb{C} \ni (q, \zeta) &\mapsto h(q, \zeta) = \{q_1 + \gamma(\zeta), \dots, q_n + \gamma(\zeta)\} \in \mathcal{C}^n(\mathbb{T}^2), \\ M_0 \times \mathbb{T}^2 \ni (q, t) &\mapsto \tilde{h}(q, t) = \{q_1 + t, \dots, q_n + t\} \in \mathcal{C}^n(\mathbb{T}^2) \end{aligned} \quad (4.5)$$

¹⁰We have also proved that $\alpha(s) = \sigma^{-1}s\sigma$

are holomorphic coverings, as well.

The following theorem completes the classification of endomorphisms of torus configuration spaces.

THEOREM 4.8. *If $m > 2$, then a holomorphic map $F: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^m(\mathbb{T}^2)$ is orbit-like if and only if it is abelian.*

Proof. Let F be abelian. By the abelianization of defining relations (2.2) - (2.7), we see that $H_1(\mathcal{C}^n(\mathbb{T}^2), \mathbb{Z}) = B_n(\mathbb{T}^2)/B'_n(\mathbb{T}^2) = \mathbb{Z}_2 \oplus \mathbb{Z}^2$. As it was already mentioned, $B_m(\mathbb{T}^2) = \pi_1(\mathcal{C}^m(\mathbb{T}^2))$ has no torsion. It follows that $\text{Im } F_*$ is either \mathbb{Z}^2 or \mathbb{Z} or trivial. Since $\pi_1(M_0)/(\pi_1(M_0))' = \mathbb{Z}_{2n}$, there is no non-trivial homomorphism $\pi_1(M_0) \rightarrow \text{Im } F_*$; that is, the homomorphism $(F \circ h)_*$ is trivial. This implies that there exists a holomorphic map $f = (f_1, \dots, f_m)$ which fits to the commutative diagram

$$\begin{array}{ccc} M_0 \times \mathbb{C} & \xrightarrow{f} & \mathcal{E}^m(\mathbb{T}^2) \\ h \downarrow & \searrow F \circ h & \downarrow p \\ \mathcal{C}^n(\mathbb{T}^2) & \xrightarrow{F} & \mathcal{C}^m(\mathbb{T}^2), \end{array}$$

where h is defined in (4.5). The homomorphism f_* of the fundamental groups induced by f is trivial. Hence, for any j , the composition

$$f_j - f_1 = (q_j - q_1) \circ f: M_0 \times \mathbb{C} \xrightarrow{f} \mathcal{E}^m(\mathbb{T}^2) \xrightarrow{q_j - q_1} \mathbb{T}^2 \setminus \{0\}$$

is contractible and lifts to a holomorphic map $g_j: M_0 \times \mathbb{C} \rightarrow \mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ into the universal covering \mathbb{D} of $\mathbb{T}^2 \setminus \{0\}$. Since $M_0 \times \mathbb{C}$ is algebraic, the Liouville's theorem shows that $g_j = \text{const}$ and, thereby, $f_j - f_1 = \text{const} = c_j \in \mathbb{T}^2 \setminus \{0\}$. Thus, $f(q) = (0 + f_1(q), c_2 + f_1(q), \dots, c_m + f_1(q))$, which shows that f is orbit-like.

Suppose now that F is orbit-like. To prove that F is abelian, it suffices to show that for any point $q \in \mathcal{C}^m(\mathbb{T}^2)$, the fundamental group of any connected component of the $(\text{Aut } \mathbb{T}^2)$ -orbit $\mathcal{O}_q = (\text{Aut } \mathbb{T}^2)(q)$ is abelian. For $m > 2$, any component of \mathcal{O}_q is diffeomorphic to the orbit \mathcal{O}_q^* of the action of \mathbb{T}^2 in $\mathcal{C}^m(\mathbb{T}^2)$ by translations. The latter orbit \mathcal{O}_q^* is a quotient group of \mathbb{T}^2 by a finite subgroup and hence is homeomorphic to \mathbb{T}^2 . Thus, $\pi_1(\mathcal{O}_q^*) = \mathbb{Z}^2$. \square

We conclude this section with two simple statements about abelian maps.

4.3. Splitting of abelian maps. Up to a homotopy, any abelian map

$$f: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^m(\mathbb{T}^2)$$

splits to a composition $g \circ s$ of the standard map $s: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathbb{T}^2$, $s(q) = q_1 + \dots + q_n$, defined in Definition 4.7 and an appropriate continuous map $g: \mathbb{T}^2 \rightarrow \mathcal{C}^m(\mathbb{T}^2)$. Thus, we obtain the commutative up to a homotopy diagram

$$\begin{array}{ccc} \mathcal{C}^n(\mathbb{T}^2) & \xrightarrow{f} & \mathcal{C}^m(\mathbb{T}^2) \\ s \searrow & & \nearrow g \\ & \mathbb{T}^2 & \end{array}$$

Indeed, let $p: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ be the standard projection and

$$\gamma = (\gamma(t), a_2, \dots, a_n) \subset \mathcal{E}^n(\mathbb{T}^2)$$

be a loop. The map $s \circ p$ carries γ to the loop $\gamma(t) + a_2 + \dots + a_n$ in \mathbb{T}^2 , which shows that $(s \circ p)_*$ is an epimorphism. Therefore, s_* is an epimorphism as well.

The homomorphism s_* splits to a composition $s_* = \phi \circ \alpha$, where $\alpha: B_n(\mathbb{T}^2) \rightarrow B_n(\mathbb{T}^2)/[B_n(\mathbb{T}^2), B_n(\mathbb{T}^2)] = \mathbb{Z}^2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}^2$ is the composition of the abelianization and the torsion eliminating map $\mathbb{Z}^2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}^2$, and $\phi: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is an epimorphism. Since every surjective endomorphism of \mathbb{Z}^2 is an isomorphism, it follows that $\phi^{-1} \circ s_* = \alpha$. Since the map f is abelian there exists a homomorphism $\beta: \mathbb{Z}^2 \rightarrow B_m(\mathbb{T}^2)$ such that $f_* = \beta \circ \alpha$. $\mathcal{C}^m(\mathbb{T}^2)$ is a $K(\pi, 1)$ space for $\pi = B_m(\mathbb{T}^2)$, which implies that there is a continuous map $g: \mathbb{T}^2 \rightarrow \mathcal{C}^m(\mathbb{T}^2)$ such that $g_* = \beta \circ \phi^{-1}$. Clearly f is homotopic to $g \circ s$, which proves our statement.

4.4. Holomorphic mappings $\mathbb{T}^2 \rightarrow \mathcal{C}^n(\mathbb{T}^2)$.

PROPOSITION 4.9. *Any holomorphic map $F: \mathbb{T}^2 \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ carries \mathbb{T}^2 to an orbit of the $\text{Aut } \mathbb{T}^2$ action in $\mathcal{C}^n(\mathbb{T}^2)$.*

Proof. F lifts to a holomorphic map f of the universal coverings and we have the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f=(f_1, \dots, f_n)} & \mathcal{E}^n(\mathbb{T}^2) \\ \gamma \downarrow & & \downarrow p \\ \mathbb{T}^2 & \xrightarrow{F} & \mathcal{C}^n(\mathbb{T}^2). \end{array}$$

Each f_j maps \mathbb{C} to \mathbb{T}^2 and, for $j > 1$, $f_j - f_1$ maps \mathbb{C} to $\mathbb{T}^2 \setminus \{0\}$. It follows that $f_j - f_1 = \text{const} = c_j \in \mathbb{T}^2 \setminus \{0\}$ and $f = (0 + f_1, c_2 + f_1, \dots, c_n + f_1)$. \square

5. BIREGULAR AUTOMORPHISMS

Here we describe all biregular automorphisms of the algebraic manifold $\mathcal{C}^n(\mathbb{T}^2)$.

LEMMA 5.1. *Any regular map $R: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \mathbb{T}^2$ is of the form*

$$R(Q) = \sum_{\mathbf{m} \in \mathfrak{M}_+} k_{\mathbf{m}} \mathbf{m}(q_1 + \dots + q_n) + c \text{ for all } Q = \{q_1, \dots, q_n\} \in \mathcal{C}^n(\mathbb{T}^2),$$

where $k_{\mathbf{m}} \in \mathbb{Z}$ and $c \in \mathbb{T}^2$.

Proof. Consider the map $r = R \circ p$, where $p: \mathcal{E}^n(\mathbb{T}^2) \rightarrow \mathcal{C}^n(\mathbb{T}^2)$ is the standard projection. By Lemma 3.14, $r(q) = \sum_{i=1}^n \sum_{\mathbf{m} \in \mathfrak{M}_+} k_{i, \mathbf{m}} \mathbf{m} q_i + c$. Since r must be invariant under the $\mathbf{S}(n)$ action, it follows that $k_{1, \mathbf{m}} = \dots = k_{n, \mathbf{m}} = k_{\mathbf{m}}$. Thus,

$$r(q_1, \dots, q_n) = \sum_{i=1}^n \sum_{\mathbf{m} \in \mathfrak{M}_+} k_{\mathbf{m}} \mathbf{m} q_i + c = \sum_{\mathbf{m} \in \mathfrak{M}_+} k_{\mathbf{m}} \mathbf{m}(q_1 + \dots + q_n) + c$$

and $R(Q) = \sum_{\mathbf{m} \in \mathfrak{M}_+} k_{\mathbf{m}} \mathbf{m}(q_1 + \dots + q_n) + c$. \square

THEOREM 5.2. *For $n > 2$, any biregular automorphism F of $\mathcal{C}^n(\mathbb{T}^2)$ is of the form $F(Q) = AQ$, where $A \in \text{Aut } \mathbb{T}^2$.*

Proof. Since for $n > 2$ the group $B_n(\mathbb{T}^2)$ is non-abelian, F is non-abelian. By Theorem 1.1 and Remark 4.6 from Section 4.2, there is a unique regular map $T: \mathcal{C}^n(\mathbb{T}^2) \rightarrow \text{Aut } \mathbb{T}^2$ such that $F(Q) = T(Q)Q$ for all $Q \in \mathcal{C}^n(\mathbb{T}^2)$.

For any $Q \in \mathcal{C}^n(\mathbb{T}^2)$, $T(Q)$ is an automorphism of \mathbb{T}^2 and, by Claim B in Section 3.2 and Lemma 5.1, it carries a point $z \in \mathbb{T}^2$ to the point

$$T(Q)z = \mathbf{m}_0 z + \sum_{\mathbf{m} \in \mathfrak{M}_+} k_{\mathbf{m}} \mathbf{m}(q_1 + \dots + q_n) + c, \quad (5.1)$$

where $\mathfrak{m}_0 \in \mathfrak{M}$, $k_{\mathfrak{m}} \in \mathbb{Z}$, and $c \in \mathbb{T}^2$ do not depend on z and Q .

Recall that $s(Q) = q_1 + \dots + q_n$ for $Q = \{q_1, \dots, q_n\}$ and set $s_1 = s \circ F$, that is,

$$s_1(Q) = (s \circ F)(Q) = s(T(Q)Q) = T(Q)q_1 + \dots + T(Q)q_n.$$

Using (5.1) for $z = q_1, \dots, q_n$, we see that

$$\begin{aligned} s_1(Q) &= \mathfrak{m}_0(q_1 + \dots + q_n) + n \left(\sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m} (q_1 + \dots + q_n) + c \right) \\ &= (\mathfrak{m}_0 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m}) (q_1 + \dots + q_n) + nc. \end{aligned} \quad (5.2)$$

On the other hand, F^{-1} is a regular automorphism as well. By the same argument, there is a unique regular $T': \mathcal{C}^n(\mathbb{T}^2) \rightarrow \text{Aut } \mathbb{T}^2$ such that $F^{-1}(Q) = T'(Q)Q$ for $Q \in \mathcal{C}^n(\mathbb{T}^2)$. As above, we conclude that $T'(Q)$ carries a point $z \in \mathbb{T}^2$ to the point

$$T'(Q)z = \mathfrak{m}'_0 z + \sum_{\mathfrak{m} \in \mathfrak{M}_+} k'_{\mathfrak{m}} \mathfrak{m} (q_1 + \dots + q_n) + c', \quad (5.3)$$

where $\mathfrak{m}'_0 \in \mathfrak{M}$, $k'_{\mathfrak{m}} \in \mathbb{Z}$, and $c' \in \mathbb{T}^2$ do not depend on z and Q . Since $s_1 \circ F^{-1} = s$, formulas (5.2)–(5.3) imply

$$\begin{aligned} q_1 + \dots + q_n &= s(Q) = s_1(F^{-1}(Q)) = s_1(\{T'(Q)q_1, \dots, T'(Q)q_n\}) \\ &= (\mathfrak{m}_0 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m}) (T'(Q)q_1 + \dots + T'(Q)q_n) + nc \\ &= (\mathfrak{m}_0 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m}) \sum_{i=1}^n (\mathfrak{m}'_0 q_i + \sum_{\mathfrak{m} \in \mathfrak{M}_+} k'_{\mathfrak{m}} \mathfrak{m} (q_1 + \dots + q_n) + c') + nc. \end{aligned}$$

By changing the order of the summation, we obtain

$$\begin{aligned} q_1 + \dots + q_n &= (\mathfrak{m}_0 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m}) (\mathfrak{m}'_0 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k'_{\mathfrak{m}} \mathfrak{m}) (q_1 + \dots + q_n) \\ &\quad + n(\mathfrak{m}_0 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m}) c' + nc. \end{aligned} \quad (5.4)$$

Since $q_1 + \dots + q_n$ runs over the whole torus, (5.4) shows that the composition $\lambda = \mu \circ \nu = \nu \circ \mu$ of the torus endomorphisms $\mu: z \mapsto (\mathfrak{m}_0 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m})z$ and

$\nu: z \mapsto (\mathfrak{m}'_0 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k'_{\mathfrak{m}} \mathfrak{m})z$ is the identity automorphism. Hence μ and ν are torus

automorphisms sending 0 to 0 and, by Claim B, $\mu(z) \equiv \mathfrak{m}_1 z$ with some $\mathfrak{m}_1 \in \mathfrak{M}$. Consequently, $(\mathfrak{m}_0 - \mathfrak{m}_1 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m})z \equiv 0$, i.e. $\mathfrak{m}_0 - \mathfrak{m}_1 + n \sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m} = 0$.

Since $n > 2$ and elements of \mathfrak{M}_+ are linearly independent over \mathbb{Q} , the latter cannot happen unless $\sum_{\mathfrak{m} \in \mathfrak{M}_+} k_{\mathfrak{m}} \mathfrak{m} = 0$. Therefore, $T(Q)z = \mathfrak{m}_0 z + c$ for all Q and z , which completes the proof. \square

6. CONFIGURATION SPACES OF UNIVERSAL FAMILIES

Here we construct configuration spaces of the universal Teichmüller family of tori and describe their holomorphic self-maps.

The Teichmüller space $T(1, 1)$ of tori with one marked point is isomorphic to the upper half plane \mathbb{H}^+ . The group $H = \mathbb{Z} \times \mathbb{Z}$ acts discontinuously and freely in the space $\mathcal{V} = T(1, 1) \times \mathbb{C} = \mathbb{H}^+ \times \mathbb{C}$ by weighted translations $(\tau, z) \mapsto (\tau, z + l + m\tau)$, $(l, m) \in H$. Let $V(1, 1) = \mathcal{V}/H$; the map $\psi: \mathcal{V} \rightarrow V(1, 1)$ is a covering and the holomorphic projection $\pi: V(1, 1) \rightarrow \mathbb{H}^+ = T(1, 1)$ is called the *universal Teichmüller family* of tori with one marked point (see [27], Sec. 4.11). All fibres $\pi^{-1}(\tau)$ are tori; each of them carries a natural group structure, marked points are neutral elements and form a holomorphic section of π .

DEFINITION 6.1. Let $\mathcal{C}_\pi^n(V(1, 1))$ be the complex subspace of the configuration space $\mathcal{C}^n(V(1, 1))$ of $V(1, 1)$ consisting of all $Q = \{q_1, \dots, q_n\} \in \mathcal{C}^n(V(1, 1))$ such that $\pi(q_1) = \dots = \pi(q_n)$. Define the holomorphic projection $\rho: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow T(1, 1)$ by $\rho(Q) = \pi(q_1) = \dots = \pi(q_n)$, $Q = \{q_1, \dots, q_n\} \in \mathcal{C}_\pi^n(V(1, 1))$; the triple $\{\rho, \mathcal{C}_\pi^n(V(1, 1)), T(1, 1)\}$, or simply $\rho: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow T(1, 1)$, is said to be the *fibred configuration space of the universal Teichmüller family* $\pi: V(1, 1) \rightarrow T(1, 1)$. A *fibred morphism* of fibred configuration spaces is a holomorphic map $F: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow \mathcal{C}_\pi^m(V(1, 1))$ which fits into the commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\pi^n(V(1, 1)) & \xrightarrow{F} & \mathcal{C}_\pi^m(V(1, 1)) \\ & \searrow \rho & \swarrow \rho \\ & T(1, 1) & \end{array} \quad (6.1)$$

One may similarly define the corresponding *ordered* fibred configuration spaces $\theta: \mathcal{E}_\pi^n(V(1, 1)) \rightarrow T(1, 1)$ and their fibred morphisms. The *fibred power*

$$\theta: (V(1, 1))_\pi^n \rightarrow T(1, 1)$$

is defined by $(V(1, 1))_\pi^n = \{(q_1, \dots, q_n) \in (V(1, 1))^n \mid \pi(q_1) = \dots = \pi(q_n)\}$ and $\theta(q_1, \dots, q_n) = \pi(q_1)$. The natural covering map $\mu: \mathcal{E}_\pi^n(V(1, 1)) \rightarrow \mathcal{C}_\pi^n(V(1, 1))$ is also a fibred morphism. The closed complex subspace $S = \{(q_1, \dots, q_n) \in (V(1, 1))_\pi^n \mid q_1 = \dots = q_n\}$ of $(V(1, 1))_\pi^n$ is called the *small diagonal* of the fibred power $\theta: (V(1, 1))_\pi^n \rightarrow T(1, 1)$; it is isomorphic to $V(1, 1)$.

REMARK 6.2. The map

$$\Psi = (\psi, \dots, \psi): (\mathbb{H}^+ \times \mathbb{C})^n \rightarrow (V(1, 1))_\pi^n$$

is a universal holomorphic covering and the preimage $\mathcal{P}_n = \Psi^{-1}((V(1, 1))_\pi^n) = \{((\tau_1, z_1), \dots, (\tau_n, z_n)) \in (\mathbb{H}^+ \times \mathbb{C})^n \mid \tau_1 = \dots = \tau_n\}$ of $(V(1, 1))_\pi^n \subset (V(1, 1))^n$ is isomorphic to $\mathbb{H}^+ \times \mathbb{C}^n$, i.e., it is irreducible and non-singular. Since Ψ is locally biholomorphic, the space $(V(1, 1))_\pi^n = \Psi(\mathcal{P}_n)$ is irreducible and non-singular and the map $\Psi|_{\mathcal{P}_n}: \mathcal{P}_n \rightarrow (V(1, 1))_\pi^n$ is a universal covering. The space $\mathcal{E}_\pi^n(V(1, 1)) \subset (V(1, 1))_\pi^n$ is the complement of a proper analytic subset of $(V(1, 1))_\pi^n$; hence $\mathcal{E}_\pi^n(V(1, 1))$ is a connected complex manifold. Since $\mu: \mathcal{E}_\pi^n(V(1, 1)) \rightarrow \mathcal{C}_\pi^n(V(1, 1))$ is a holomorphic covering, $\mathcal{C}_\pi^n(V(1, 1))$ also is a connected complex manifold.

Notice that the preimage $\mathcal{F} = \Psi^{-1}(\mathcal{E}_\pi^n(V(1, 1)))$ of $\mathcal{E}_\pi^n(V(1, 1)) \subset (V(1, 1))_\pi^n$ is an open dense subset of $\mathcal{P}_n \cong \mathbb{H}^+ \times \mathbb{C}^n$ and the maps $\Psi|_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{E}_\pi^n(V(1, 1))$ and $\mu \circ \Psi|_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{C}_\pi^n(V(1, 1))$ are holomorphic coverings.

DEFINITION 6.3. Let $g: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow V(1, 1)$ be a fibred morphism. Any point $Q \in \mathcal{C}_\pi^n(V(1, 1))$ belongs to a certain fibre $\rho^{-1}(\tau)$, which is the configuration space $\mathcal{C}^n(\pi^{-1}(\tau))$ of the torus $\mathbb{T}_\tau^2 = \pi^{-1}(\tau)$; so Q may be viewed as an n -point subset of \mathbb{T}_τ^2 . Since g is a fibred morphism, $g(Q)$ is a point of the same torus \mathbb{T}_τ^2 ; thus,

$Q + g(Q)$ and $-Q + g(Q)$ are well-defined n -point subsets of \mathbb{T}_τ^2 , or, which is the same, points of $\mathcal{C}^n(\mathbb{T}_\tau^2) \subset \mathcal{C}_\pi^n(V(1, 1))$. This provides us with two fibred maps $G_\pm = \pm \text{Id} + g: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow \mathcal{C}_\pi^n(V(1, 1))$ defined by $Q \mapsto \pm Q + g(Q)$.

LEMMA 6.4. *The fibred maps G_\pm are holomorphic.*

Proof. There is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{U} & & \\
 \downarrow \lambda & \searrow h & \\
 \mathcal{F} & & T(1, 1) \times \mathbb{C} \\
 \downarrow \varphi = \mu \circ \Psi & & \downarrow \psi \\
 \mathcal{C}_\pi^n(V(1, 1)) & \xrightarrow{g} & V(1, 1) \\
 \searrow \rho & & \swarrow \pi \\
 & T(1, 1), &
 \end{array} \tag{6.2}$$

where $\lambda: \mathcal{U} \rightarrow \mathcal{F}$ is the pull-back of the covering $\psi: T(1, 1) \times \mathbb{C} \rightarrow V(1, 1)$ along the map $g \circ \varphi: \mathcal{F} \rightarrow V(1, 1)$ and r , as usual, is the restriction to $\mathcal{U} \subset \mathcal{F} \times T(1, 1) \times \mathbb{C}$ of the projection $\mathcal{F} \times T(1, 1) \times \mathbb{C} \rightarrow T(1, 1) \times \mathbb{C}$. By the construction, h is a fibred morphism. Since \mathcal{F} is a domain in $T(1, 1) \times \mathbb{C}^n$, it follows that $\lambda(u) = (\rho \circ \varphi \circ \lambda(u), z_1(u), \dots, z_n(u))$ for $u \in \mathcal{U}$, where for all $i = 1, \dots, n$ the functions $z_i: \mathcal{U} \rightarrow \mathbb{C}$ are holomorphic. Denote the natural projection $T(1, 1) \times \mathbb{C} \rightarrow \mathbb{C}$ by z . Define the map $R = \text{Id} + h: \mathcal{U} \rightarrow \mathcal{P}_n$ by $R(q) = (\rho \circ \varphi \circ \lambda(u), z_1(u) + z(h(u)), \dots, z_n(u) + z(h(u)))$ for $q \in \mathcal{U}$. Since $\psi(\rho \circ \varphi \circ \lambda(u), z_i(u)) \neq \psi(\rho \circ \varphi \circ \lambda(u), z_j(u))$ for any $u \in \mathcal{U}$ and $i \neq j$, it follows that for all $m, n \in \mathbb{Z}$ we have $z_i(u) - z_j(u) \neq m + n\rho(\varphi(\lambda(u)))$ and $(z_i(u) + z(h(u))) - (z_j(u) + z(h(u))) \neq m + n\rho(\varphi(\lambda(u)))$; thus $R(u) \in \mathcal{F}$, that is, $R(\mathcal{U}) \subset \mathcal{F}$. Set $R' = \varphi \circ R: \mathcal{U} \rightarrow \mathcal{C}_\pi^n(V(1, 1))$.

Let us show that for $u \in (\varphi \circ \lambda)^{-1}(Q)$ we have $R'(u) = Q + g(Q)$ and the map R' induces the well-defined single-valued fibred morphism which is equal to $G_+: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow \mathcal{C}_\pi^n(V(1, 1))$. Indeed, for $u \in (\varphi \circ \lambda)^{-1}(Q)$ we have $R(u) = (\rho(Q), z_1(u) + z(h(u)), \dots, z_n(u) + z(h(u)))$. Notice that for $\tau \in T(1, 1)$ and $x, y \in \mathbb{C}$ we have $\psi(\tau, x + y) = \psi(\tau, x) + \psi(\tau, y)$, where the latter summation is the sum in the torus $\mathbb{T}_\tau^2 = \psi^{-1}(\tau)$. Let $\tau = \rho(Q)$ and $\zeta = z(h(u))$. Therefore

$$\begin{aligned}
 R'(u) &= \varphi(R(u)) = \mu(\Psi(R(u))) = \mu(\Psi(\tau, z_1(u) + \zeta, \dots, z_n(u) + \zeta)) \\
 &= \mu(\psi(\tau, z_1(u) + \zeta), \dots, \psi(\tau, z_n(u) + \zeta)) \\
 &= \mu(\psi(\tau, z_1(u)) + \psi(\tau, \zeta), \dots, \psi(\tau, z_n(u)) + \psi(\tau, \zeta)) \\
 &= \mu(\psi(\tau, z_1(u)), \dots, \psi(\tau, z_n(u))) + \psi(\tau, \zeta) \\
 &= \mu(\Psi((\tau, z_1(u)), \dots, (\tau, z_n(u)))) + \psi(\tau, \zeta) \\
 &= \varphi(\tau, z_1(u), \dots, z_n(u)) + \psi(\tau, \zeta) = Q + g(Q).
 \end{aligned}$$

The same argument shows that $G_- = -\text{Id} + g$ is also holomorphic. \square

REMARK 6.5. Similarly to the Definition 6.3, for any fibred morphism

$$f = (f_1, \dots, f_n): \mathcal{E}_\pi^n(V(1, 1)) \rightarrow \mathcal{E}_\pi^n(V(1, 1))$$

one can define fibred morphisms

$$\text{Id} + f: \mathcal{E}_\pi^n(V(1, 1)) \rightarrow (V(1, 1))_\pi^n$$

and

$$-\text{Id} + f: \mathcal{E}_\pi^n(V(1, 1)) \rightarrow (V(1, 1))_\pi^n$$

which take any point $q = (q_1, \dots, q_n) \in \mathcal{E}_\pi^n(V(1, 1))$ respectively to

$$q + f(q) = (q_1 + f_1(q), \dots, q_n + f_n(q)) \in (V(1, 1))_\pi^n$$

and to

$$-q + f(q) = (-q_1 + f_1(q), \dots, -q_n + f_n(q)) \in (V(1, 1))_\pi^n.$$

The following theorem is an analogue of Theorem 1.1 for the case of the fibred morphisms.

THEOREM 6.6. *Let $n > 4$. For any fibred non-abelian morphism $F: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow \mathcal{C}_\pi^n(V(1, 1))$, there is a fibred morphism $g: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow V(1, 1)$ such that F is either $\text{Id} + g$ or $-\text{Id} + g$.*

Proof. According to Theorem 1.1, for any $\tau \in T(1, 1)$ there exists a unique holomorphic map $T_\tau: \rho^{-1}(\tau) \rightarrow \text{Aut } \pi^{-1}(\tau)$ such that $F(Q) = T_\tau(Q)Q$ for any $Q \in \rho^{-1}(\tau) \subset \mathcal{C}_\pi^n(V(1, 1))$. There are no complex multiplication on a generic torus. Thus, for any generic $\tau \in T(1, 1)$ and any $Q \in \rho^{-1}(\tau)$, there exists $c_\tau(Q) \in \pi^{-1}(\tau)$ such that the automorphism $T_\tau(Q)$ maps a point $z \in \pi^{-1}(\tau)$ either to $z + c_\tau(Q)$ or to $-z + c_\tau(Q)$; notice that the representation is unique. By Baire Category Theorem, there exists an open set $D \subset T(1, 1)$ and its dense subset $D' \subset D$ such that either (*) for all $\tau \in D'$ we have $T_\tau(Q) = z + c_\tau(Q)$ or (**) for all $\tau \in D'$ we have $T_\tau(Q) = -z + c_\tau(Q)$. Without loss of generality, we may assume the case (**), that is, for all $\tau \in D'$ we have $T_\tau(Q) = -z + c_\tau(Q)$. By Theorem 1.4, $F: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow \mathcal{C}_\pi^n(V(1, 1))$ can be lifted to a strictly equivariant fibred morphism $f: \mathcal{E}_\pi^n(V(1, 1)) \rightarrow \mathcal{E}_\pi^n(V(1, 1))$ commuting with $\mathbf{S}(n)$ -action (see the end of the proof of Theorem 4.4). The condition (**) implies that the fibred morphism $h = \text{Id} + f: \mathcal{E}_\pi^n(V(1, 1)) \rightarrow (V(1, 1))_\pi^n$ maps the set $\theta^{-1}(D') \subset \mathcal{E}_\pi^n(V(1, 1))$ into the closed complex subspace $S \subset (V(1, 1))_\pi^n$ (see Definition 6.1). The morphism h is continuous and $\theta^{-1}(D')$ is dense in $\theta^{-1}(D)$, thus $h(\theta^{-1}(D)) \subseteq S$. Since $\mathcal{E}_\pi^n(V(1, 1))$ is irreducible (see Remark 6.2), it follows that $h(\mathcal{E}_\pi^n(V(1, 1))) \subseteq S \cong V(1, 1)$. By the definition of h , it commutes with $\mathbf{S}(n)$ -action, thus, h is $\mathbf{S}(n)$ -invariant and it induces a fibred morphism $g: \mathcal{C}_\pi^n(V(1, 1)) \rightarrow V(1, 1)$ with the desired properties. \square

REMARK 6.7. If F is an automorphism, the statement of the above theorem holds true for $n = 3, 4$.

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